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DEPARTMENT OF CIVIL ENGINEERING
AND ENGINEERING MECHANICS
INSTITUTE OF AIR FLIGHT STRUCTURES

THREE-DIMENSIONAL AND SHELL THEORY ANALYSIS OF AXIALLY-SYMMETRIC MOTIONS OF CYLINDERS

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Office of Scientific Research
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Technical Note No.1
CU-3-55-AF-1247-CE
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ABSTRACT

The frequency (or phase velocity) of axially symmetric free vibrations in an elastic, isotropic circular cylinder of medium thickness is studied on the basis of the three-dimensional linear theory of elasticity and also on the basis of several different shell theories. To be in good agreement with the solution of the three-dimensional equations for short wave lengths, an approximate theory has to include the influence of rotatory inertia and transverse shear deformation for example in a manner similar as in Mindlin's plate theory. A shell theory of this (Timoshenko) type is deduced from the three-dimensional elasticity theory. From a comparison of phase velocities it appears that, to a good approximation, membrane and curvature effects on one hand and flexural, rotatory inertia and shear deformation effects, on the other hand, are mutually exclusive in two ranges of wave lengths, separated by a "transition" wave length. Thus, in the full range of wave lengths, the associated lowest phase velocities may be determined on the basis of the membrane shell theory (for wave lengths larger than the transition wave length) and on the basis of Mindlin's plate theory (for wave lengths smaller than the transition wave length).

INTRODUCTION

Due to a rapidly increasing technological importance, considerable attention has been given during recent years to the analysis of shells and the literature on shell theory grows at an accelerated rate. In the course of reconsideration and refinement of the existing theories, in particular that of Love (1)[1], various objections have been raised as to the method of deriving a shell theory and also as to the resulting equations. It was emphasized by Vlasov (2), for example, that certain shell theories are in contradiction to the principle of conservation of energy and, also, that the approximations, introduced in the course of the development, are sometimes dealt with subsequently in a non-consistent manner.

Systematic procedures for deducing shell theories of various degree of accuracy, starting from three-dimensional equations, were discussed by a variety of authors, for example by Hildebrand, Reissner and Thomas (3) and by Kennard (4, 5), the latter basing his development on the work of Epstein (6). Several such theories have been compared with one another recently by Naghdi and Berry (7) in a qualitative manner, but the authors, in their own judgement, did not succeed in arriving at definite conclusions concerning the relative merits of the different approximations used by various authors in the classical theory of shells.

Thus, the state of knowledge, briefly described above, definitely calls for the establishment of criteria for the range of applicability of the various shell theories. Solutions obtained by different shell theories

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[1] Numbers in parentheses refer to the Bibliography at the end of the paper.

should be compared not only on a relative basis, but also with solutions from the three-dimensional linear theory of elasticity.

Reviewing recently a controversial paper by Hwang (8) on the subject of shell theories, where the correctness of various approaches was questioned, Truesdell (9) appraised the situation by stating that in his opinion "all work on this subject is purely formal, and the various results obtained by different perturbation processes cannot be shown to be right or wrong by the a priori arguments always employed. What is lacking is a mathematical theorem making precise the status of solutions of any given set of proposed equations with respect to corresponding solutions of the three-dimensional theory."

The present paper is intended to be a first contribution in this direction. In it, no precise criteria or theorems are established; however certain quantitative though limited information is deduced. Comparing, for the first time, solutions of the three-dimensional theory with solutions of shell theories it was natural to start out by considering a simple a problem as possible. As far as the geometry of the shell is concerned, a circular cylindrical shell was selected, and since the authors were interested in the dynamic behavior of the shell, free, axially symmetric vibrations (or waves) were studied. Furthermore, it was decided to test merely several field equations, such that the shell was considered as being infinite. The important and typical question of the edge effect in shell theory was thus not raised at present at all.

An analysis of traveling waves in a circular cylinder on the basis of the three-dimensional theory was carried out by Fay (10), but his results were not used, because it was noticed that an error has slipped into his development.

Thus, the derivation of the frequency equation is presented briefly in the first section of this paper. A more complete discussion of this three-dimensional theory, including the transition to the two limiting cases of a flat plate (Rayleigh-Lamb solution (11, 12)) and of a solid rod (Pochhammer solution (13)), as well as the question of parasitic modes found by Fay (10), will be taken up by the present authors in a separate paper.

Free motions in a cylinder using shell theories were analyzed by Rayleigh (14), and more recently by Baron and Bleich (15) and by Junger and Rosato (16), but since only membrane effects or, in addition, bending effects were taken into account, the results cannot be expected to be valid for the full frequency range, that is for very short wave lengths.

Being guided by the work of Mindlin (17) in the theory of plates, which includes in addition to the classical terms (flexural stiffness and transverse inertia), terms accounting for transverse shear deformation and rotatory inertia, an analogous shell theory was deduced for axially symmetric motions in a cylinder. Thus, in addition to the terms of the classical shell theory, (compressional and flexural stiffness, transverse and longitudinal inertia), the equations derived presently include the effect, of transverse shear deformation, introduced previously into more general shell equations of equilibrium by Hildebrand, Reissner and Thomas (3), and the effect of rotatory inertia.

The effect of these various terms on the velocity of traveling waves in a cylinder is analyzed and in each case the transition to a flat plate is performed. The "shear deflection coefficient," appearing in the relation between average transverse shear stress and strain was determined in two ways analogous to Mindlin (17).

Computations of the lowest wave velocity as a function of the wave length were made for a shell of thickness to radius ratio of 1/30. It was found that excellent agreement between the approximate and the exact solution may be reached for the full frequency range, by including in a shell theory all the effects mentioned above.

Two fairly distinct types of shell behavior were established. For wave lengths larger than a certain "transition" (relative) length, the compressional stiffness is predominant and the membrane theory gives good results. For wave lengths smaller than this transition length, the shell behaves essentially like a plate and, for good results, calls for inclusion of transverse shear deformation and rotatory inertia effects. Thus, the use of bending theory of shells appears to be of little value. In the full frequency range, vibrations of a shell may be described with good accuracy by the membrane theory and by Mindlin's plate theory. It may be expected, that this statement will not be valid for non-axially symmetric motions, which will be studied by the present authors in a forthcoming paper.

Since interest concentrated at present on free vibration solutions, no attention was given to terms arising from external loading and no energy functions were constructed, which would permit the establishement of appropriate initial and boundary conditions, sufficient to assure a unique solution. Furthermore, attention was restricted to the lower mode only and no mode shapes were investigated. It is intended, however, to study these questions, when dealing with a more general theory of cylindrical shells, valid for non axially-symmetric motions.

SOLUTION BY THE THREE-DIMENSIONAL THEORY

The circular cylindrical shell of inner radius a and outer radius b, is referred to cylindrical coordinates $r, \theta, x,$ the x-axis coinciding with the axis of the shell. Designating by u_r, u_θ, u_x the components of the displacement in this coordinate system, attention will be given to axially symmetric motions only, that is to

$$u_\theta = 0 \quad , \quad \frac{\partial}{\partial \theta} = 0 \qquad [1]$$

The linear elasticity equations of motion, for a homogeneous, isotropic shell material obeying Hooke's law, may then be given in the form

$$\begin{aligned}(\lambda+2\mu)\frac{\partial \Delta}{\partial r} + 2\mu\frac{\partial \omega_\theta}{\partial x} &= \rho\ddot{u}_r \\ (\lambda+2\mu)\frac{\partial \Delta}{\partial x} - \frac{2\mu}{r}\frac{\partial}{\partial r}(r\omega_\theta) &= \rho\ddot{u}_x\end{aligned} \qquad [2]$$

where the dilation Δ is

$$\Delta = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{\partial u_x}{\partial x} \qquad [3]$$

and the rotation ω_θ

$$\omega_\theta = \frac{1}{2}\left(\frac{\partial u_r}{\partial x} - \frac{\partial u_x}{\partial r}\right) \qquad [4]$$

λ, μ are Lame's constants and ρ is the mass density. Dots indicate differentiation with respect to time, t.

Since the shell is assumed to be free from surface tractions, the boundary conditions, to be imposed on the cylindrical surface of the shell, are

$$\text{For } r = a, b: \quad \sigma_{rr} = \sigma_{rx} = 0 \qquad [5]$$

In solving the equations of motion [2], it is found more convenient to consider Δ and ω_θ as the dependent variables, rather than u_r and u_x. The equations [2] take then the form

$$\frac{\partial^2 \Delta}{\partial r^2} + \frac{1}{r}\frac{\partial \Delta}{\partial r} + \frac{\partial^2 \Delta}{\partial x^2} = \frac{\rho}{\lambda + 2\mu}\frac{\partial^2 \Delta}{\partial t^2}$$
$$\frac{\partial^2 \omega_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial \omega_\theta}{\partial r} - \frac{\omega_\theta}{r^2} + \frac{\partial^2 \omega_\theta}{\partial x^2} = \frac{\rho}{\mu}\frac{\partial^2 \omega_\theta}{\partial t^2} \qquad [6]$$

Assuming the motion to be harmonic

$$\Delta = \bar{\Delta} e^{i(\omega t - \alpha x)}$$
$$\omega_\theta = \bar{\omega}_\theta e^{i(\omega t - \alpha x)} \qquad [7]$$

where ω is the circular frequency and α the phase function, related to the phase velocity C by

$$\alpha = \omega / C \qquad [8]$$

the equations of motion [6] become

$$\frac{\partial^2 \bar{\Delta}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\Delta}}{\partial r} + \beta^2 \bar{\Delta} = 0$$
$$\frac{\partial^2 \bar{\omega}_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\omega}_\theta}{\partial r} - \frac{\bar{\omega}_\theta}{r^2} + \gamma^2 \bar{\omega}_\theta = 0 \qquad [9]$$

where

$$\beta^2 = \omega^2\left(\frac{1}{C_c^2} - \frac{1}{C^2}\right)$$
$$\gamma^2 = \omega^2\left(\frac{1}{C_s^2} - \frac{1}{C^2}\right) \qquad [10]$$

and C_c, C_s are the phase velocities of compressional and shear waves, respectively

$$C_s^2 = \mu/\rho \qquad C_c^2 = \frac{\lambda + 2\mu}{\rho} \qquad [11]$$

Equations [9] are of the Bessel type, and $\bar{\Delta}$ and $\bar{\omega}_0$ must be taken proportional to the Bessel functions $J_0(\beta r)$, $Y_0(\beta r)$ and $J_1(\gamma r)$, $Y_1(\gamma r)$, respectively. The displacements u_r, u_x themselves, in view of the relations [3] and [4], become therefore

$$u_r = -\left\{\beta\left[A J_1(\beta r) + B Y_1(\beta r)\right] + \alpha\left[C J_1(\gamma r) + D Y_1(\gamma r)\right]\right\} e^{i(\omega t - \alpha x)} \qquad [12]$$

$$u_x = \left\{i\gamma\left[C J_0(\gamma r) + D Y_0(\gamma r)\right] - i\alpha\left[A J_0(\beta r) + B Y_0(\beta r)\right]\right\} e^{i(\omega t - \alpha x)}$$

where A, B, C and D are arbitrary constants of integration.

The boundary conditions [5] in terms of the displacements will be

For $r = a, b$:

$$\begin{aligned} \lambda\Delta + 2\mu \frac{\partial u_r}{\partial r} &= 0 \\ \mu\left(\frac{\partial u_r}{\partial x} + \frac{\partial u_x}{\partial r}\right) &= 0 \end{aligned} \qquad [13]$$

These four boundary conditions yield four homogeneous, algebraic equations in the constants A, B, C and D. For a non-trivial solution, the vanishing of their determinant is required, which results in the characteristic equation

$$\begin{aligned} f(K) = {} & \left[K_{10}(\beta)K_{01}(\gamma) + K_{01}(\beta)K_{10}(\gamma) + (8/\pi^2\beta\gamma ab)\right. \\ & \left. + F K_{11}(\gamma)K_{00}(\beta) + (1/F) K_{11}(\beta)K_{00}(\gamma)\right] \\ & - \left[(1+\bar{B})/\gamma ab\right]\left[a K_{10}(\beta) + b K_{01}(\beta)\right] K_{11}(\gamma) \\ & - \left[(1+\bar{B})/F\gamma ab\right]\left[a K_{10}(\gamma) + b K_{01}(\gamma)\right] K_{11}(\beta) \\ & + \left[(1+\bar{B})^2/F\gamma^2 ab\right] K_{11}(\beta) K_{11}(\gamma) \\ = {} & 0 \end{aligned} \qquad [14]$$

where

$$
\begin{aligned}
K_{mn}(z) &= J_m(zb)\,Y_n(za) - J_n(za)\,Y_m(zb) \\
\beta^2 &= \alpha^2\left(\frac{c^2}{c_c^2} - 1\right) \\
\gamma^2 &= \alpha^2\left(\frac{c^2}{c_s^2} - 1\right) \\
\bar{B} &= \frac{c^2}{2c_s^2} - 1 \\
F &= \alpha^2 \bar{B}^2 / \beta\gamma
\end{aligned}
\qquad [15]
$$

It may be of interest to note that the last three terms of the characteristic equation [14] appear to have been omitted by Fay (10).

For computational purposes, the equation [14] is written conveniently in two different forms, depending upon $\frac{c}{c_s} \gtrless 1$. In the range $\frac{c}{c_s} < 1$ the arguments of the Bessel functions become imaginary and equation [14] takes the form

$$
\begin{aligned}
f(M) = {} & \Big[M_{10}(\bar\beta) M_{01}(\bar\gamma) + M_{01}(\bar\beta) M_{10}(\bar\gamma) - (2/\bar\beta\bar\gamma ab) \\
& \quad - \bar{F} M_{11}(\bar\gamma) M_{00}(\bar\beta) - (1/\bar{F}) M_{11}(\bar\beta) M_{00}(\bar\gamma) \Big] \\
& - \left[(1+\bar{B})/\bar\gamma ab\right]\left[a M_{10}(\bar\beta) - b M_{01}(\bar\beta)\right] M_{11}(\bar\gamma) \\
& + \left[(1+\bar{B})/\bar{F}\bar\gamma ab\right]\left[a M_{10}(\bar\gamma) - b M_{01}(\bar\gamma)\right] M_{11}(\bar\beta) \\
& + \left[(1+\bar{B})^2/\bar{F}\bar\gamma^2 ab\right] M_{11}(\bar\beta) M_{11}(\bar\gamma) \\
= {} & 0
\end{aligned}
\qquad [16]
$$

where

$$M_{mn}(z) = I_m(zb)K_n(za) - (-1)^{m-n} I_n(za)K_m(zb)$$
$$\bar{\beta}^2 = -\beta^2$$
$$\bar{\gamma}^2 = -\gamma^2 \qquad [17]$$
$$\bar{F} = \bar{B}^2\alpha^2/\bar{\beta}\bar{\gamma}$$

$I_n(z)$ and $K_n(z)$ are the Modified Bessel Functions of the first and second kind, in the notation of Watson (18).

In the range $1 < \frac{c}{c_s} < \frac{c}{c_c}$, β is imaginary and γ is real. Equation [14] may then be written in the form

$$\begin{aligned} f(K,M) = \Big[& M_{10}(\bar{\beta})K_{01}(\gamma) - M_{01}(\bar{\beta})K_{10}(\gamma) + (4/\pi\bar{\beta}\gamma ab) \\ & - \bar{\bar{F}} M_{00}(\bar{\beta})K_{11}(\gamma) + (1/\bar{\bar{F}})M_{11}(\bar{\beta})K_{00}(\gamma)\Big] \\ & - \big[(1+\bar{B})/\gamma ab\big]\big[aM_{10}(\bar{\beta}) - bM_{01}(\bar{\beta})\big]K_{11}(\gamma) \\ & - \big[(1+\bar{B})/\bar{\bar{F}}\gamma ab\big]\big[aK_{10}(\gamma) + bK_{01}(\gamma)\big]M_{11}(\bar{\beta}) \\ & + \big[(1+\bar{B})^2/\bar{\bar{F}}\gamma^2 ab\big] M_{11}(\bar{\beta})K_{11}(\gamma) \\ = {} & 0 \end{aligned} \qquad [18]$$

where

$$\bar{\bar{F}} = \bar{B}^2\alpha^2/\bar{\beta}\gamma \qquad [19]$$

Solving equations [16] and [18], respectively, the phase velocity $S = \frac{c}{c_s}$ versus $\delta = \frac{h}{L}$ (with $\nu = 0{\cdot}3$, $\frac{h}{R} = \frac{1}{30}$) is plotted as curve 1 in Fig. 1.

For sufficiently thin shells, the arguments of the Bessel functions appearing in the characteristic equation [16] become very large and a simplification can be made by expanding these functions asymptotically. In this case the frequency equation appears in the much simpler form:

$$
\begin{aligned}
&2\left[\cosh\bar{\beta}h\cosh\bar{\gamma}h \quad - 1\right] \\
&-\left(\bar{F}+\frac{1}{\bar{F}}\right)\sinh\bar{\beta}h\,\sinh\bar{\gamma}h \\
&+\left[(1+\bar{B})^2/\bar{F}\bar{\gamma}^2ab\right]\sinh\bar{\beta}h\,\sinh\bar{\gamma}h \\
&+\left[(1+\bar{B})h/\bar{\gamma}ab\right]\sinh\bar{\gamma}h\cosh\bar{\beta}h \\
&-\left[(1+\bar{B})h/\bar{F}\bar{\gamma}ab\right]\sinh\bar{\beta}h\cosh\bar{\gamma}h \quad = 0
\end{aligned}
\qquad [20]
$$

where h is the thickness of the shell,

Limiting Velocities

For very short wave lengths, $h\alpha\to\infty$, thus $\bar{\beta}\to\infty$, $\bar{\gamma}\to\infty$, and for very large arguments sinh z $\sim$cosh z. Hence the frequency equation [20] takes the form

$$2 - \left(\bar{F}+\frac{1}{\bar{F}}\right) = 0$$

that is

$$\bar{F} = 1$$

or

$$\sqrt{(n^2-s^2)(1-s^2)} = n\left(\tfrac{1}{2}s^2-1\right)^2, \quad 0 < s < 1 \qquad [21]$$

where

$$s = \frac{c}{c_s}$$

$$n^2 = \frac{c_c^{\,2}}{c_s^2} = 2(1-\nu)/(1-2\nu) \qquad [22]$$

Expression [21] is the equation for the velocity of Rayleigh surface waves. Thus, very short waves are propagated in the shell with the same velocity as in a plate, the latter case having been investigated by Lamb (12).

For very long wave lengths, $h\alpha \to 0$ and hence, in the limit, $\beta \to 0$, $\gamma \to 0$. It may be seen that in this case $\frac{c}{c_s} > 1$. The arguments of the Bessel functions become then very large and on expressing these functions again in asymptotic form, the frequency equation [14] reduces in the limit to

$$\left(\frac{c^2}{c_c^2} - 1\right)\left(\bar{B} - 1\right) - 2\bar{B}^2 = 0, \qquad \frac{c}{c_s} > 1$$

or

$$\frac{c^2}{c_s^2} = 2(1+\nu) \qquad [23]$$

that is

$$c^2 = E/\rho$$

which is the "bar" velocity, as it is well known from elementary bar theory and as it was determined by Pochhammer (13) for the analogous limiting case in his study of motions of solid cylinders.

Thickness-Shear Motion

If the displacement components, in addition to assumptions [1], are restricted still further by requiring

$$u_r = 0, \qquad \frac{\partial}{\partial x} = 0 \qquad [24]$$

the possible motion is designated as thickness-shear motion (17). The equations of motion [2] simplify then to

$$c_s^2\left[\frac{\partial^2 u_x}{\partial r^2} + \frac{1}{r}\frac{\partial u_x}{\partial r}\right] = \ddot{u}_x \qquad [25]$$

Assuming the motion to be periodic

$$u_x(r,t) = F(r)e^{i\omega t} \qquad [26]$$

equation [25] reduces to

$$\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} + \frac{\omega^2}{C_s^2}F = 0 \qquad [27]$$

which is a Bessel equation of zero order. Imposing boundary conditions analogous to [13] (traction-free surfaces), the resulting frequency equation may be put into the form

$$\frac{J_1(\bar{\lambda}k)}{Y_1(\bar{\lambda}k)} = \frac{J_1(\bar{\lambda})}{Y_1(\bar{\lambda})} \qquad [28]$$

where

$$k = a/b$$
$$\bar{\lambda} = \omega b/c_s \qquad [29]$$

Thus, for a given value of k , $\bar{\lambda}$ can be determined and hence ω.

For thin shells, the arguments of Bessel functions in equation [28] become large and the frequency may be computed from the simplified equation

$$\tan \bar{\lambda}(1-k) = 0 \qquad [30]$$

which was obtained after representing the Bessel functions by asymptotic expansions. The lowest frequency from [30] is

$$\omega = \pi \frac{c_s}{h} \qquad [31]$$

which is identical to the one obtained in studying the corresponding motion of a flat plate, see (12) and also (17).

APPROXIMATE SHELL THEORIES

In the theory of shells, it is customary to employ a coordinate system, formed by two axes being in the middle surface of the shell and the third axis being normal to the middle surface. Thus, the coordinates will now be taken as x, θ, z, x being still in the axial direction of the cylindrical shell and the new z-coordinate being

$$z = r - R \qquad [32]$$

where $R = \frac{a+b}{2}$ is the mean radius of the shell.

Stress-Equations of Motion

For the case of axial symmetry, the stress equations of motion may then be written as

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} + \frac{\sigma_{xz}}{R+z} = \rho \ddot{u}_x \qquad [33]$$

$$\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{xz}}{\partial x} + \frac{\sigma_{zz} - \sigma_{\theta\theta}}{R+z} = \rho \ddot{u}_z$$

It is now assumed, that the displacements u_x and u_z of the three-dimensional theory may be approximated by displacements $\bar{u}_x$ and $\bar{u}_z$, whose dependence through the thickness h of the shell shall be specified in the form

$$\bar{u}_x = u(x,t) + z\,\psi_x(x,t)$$
$$\bar{u}_z = w(x,t) \qquad [34]$$

The shell stress-equations of motion are obtained by substituting in equations [33] u_x, u_z by $\bar{u}_x$, $\bar{u}_z$, respectively, multiplying the first and the second by $(R+z)$ and the second of equations [33] by $z(R+z)$ and integrating through the shell thickness h, from $z = -\frac{h}{2}$ to $z = \frac{h}{2}$.

The justification for the multiplication mentioned above stems from energy considerations which shall be discussed by the authors elsehwere.

The resulting equations are:

$$\begin{aligned} M_{xx}' - Q_x &= \rho\frac{h^3}{12}\left(\ddot{\psi}_x + \frac{1}{R}\ddot{u}\right) \\ Q_x' - \frac{1}{R}N_{\theta\theta} &= \rho h\ddot{w} \\ N_{xx}' &= \rho h\left(\ddot{u} + \frac{h^2}{12R}\ddot{\psi}_x\right) \end{aligned} \qquad [35]$$

where use has been made of the fact that the shell is free from surface tractions, i.e. $\sigma_{xz} = \sigma_{zz} = 0$ on $z = \pm\frac{h}{2}$. Primes indicate differentiation with respect to x.

The stress resultants (or shell stresses) are defined by the following integrals:

$$\begin{aligned} N_{xx} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx}\left(1+\frac{z}{R}\right)dz \\ N_{\theta\theta} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\theta\theta}\, dz \\ Q_x &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xz}\left(1+\frac{z}{R}\right)dz \\ M_{xx} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx}\left(1+\frac{z}{R}\right)z\, dz \end{aligned} \qquad [36]$$

In case of equilibrium, i.e. if all the time derivatives vanish, the equations [35] represent a special case of a more general shell theory, deduced by Hildebrand, Reissner and Thomas (3). If the radius (of curvature) R tends to infinity, that is, the shell approaches a flat plate, the first two equations of the set [35] reduce to those derived by Mindlin (17), which govern flexural motions of plates, including the influence of rotatory inertia and shear. The third equation represents longitudinal motions of the plate, which are not coupled to flexural motions. The terms with ρh^3 represent the influence of rotatory inertia. It is seen, that there are altogether three such terms. The term $\rho h\ddot{w}$ represents transverse inertia and the term $\rho h\ddot{u}$ is longitudinal inertia.

Stress-Displacement Relations

The relation between the shell stresses defined by the set [36] and shell displacements introduced by [34], will be derived, starting from the stress-strain relations of an isotropic three-dimensional solid obeying Hooke's law. If the relation for σ_{zz} is solved for e_{zz} and then substituted into the relations for the other two normal stress components, the result is

$$\begin{aligned} \sigma_{xx} &= \frac{E}{(1-\nu^2)}\left(e_{xx} + \nu e_{\theta\theta}\right) + \frac{\nu}{1-\nu}\,\sigma_{zz} \\ \sigma_{\theta\theta} &= \frac{E}{(1-\nu^2)}\left(e_{\theta\theta} + \nu e_{xx}\right) + \frac{\nu}{1-\nu}\,\sigma_{zz} \end{aligned} \qquad [37]$$

where E is Young's modulus and ν Poisson's ratio. This procedure is justified by observing that e_{zz} would vanish on the basis of the form given to the transverse displacement w. On the other hand, unrestrained effect of Poisson's ratio will take place in the actual shell in the z-direction. These two mutually exclusive requirements may be reconciled by eliminating e_{zz} from the stress-strain relations in the manner described above.

In addition to the normal stresses, the shear stress σ_{xz} is needed, which is related to the shear deformation γ_{xz} by

$$\sigma_{xz} = G\,\gamma_{xz} \qquad [38]$$

where G is the shear modulus. With the aid of the strain displacement relations

$$\begin{aligned} e_{xx} &= \frac{\partial u_x}{\partial x} \\ e_{\theta\theta} &= u_z/(R+z) \\ \gamma_{xz} &= \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \end{aligned} \qquad [39]$$

the approximate strains are evaluated using assumptions [34] and substituted into the stress-strain relations [37], [38]. Then, integrations are carried out as specified by the forms [36]. The results are modified in three respects:

1. The integrals containing σ_{zz} are omitted, just as in the classical beam or plate theories.

2. The integral containing γ_{xz} is multiplied by a constant κ^2 (the shear coefficient), to be determined later.

3. Logarithmic terms occuring in the integration are expanded in series of ascending powers of $\frac{h}{R}$ and terms higher than the cubic one are neglected.

The resulting shell stress-displacement relations are

$$
\begin{aligned}
N_{xx} &= \frac{Eh}{(1-\nu^2)}\left[u' + \nu\frac{w}{R} + \frac{h^2}{12R}\psi_x'\right] \\
N_{\theta\theta} &= \frac{Eh}{(1-\nu^2)}\left[\nu u' + \frac{w}{R}\left(1+\frac{h^2}{12R^2}\right)\right] \\
Q_x &= \kappa^2 Gh\left[\psi_x + w'\right] \\
M_{xx} &= D\left[\psi_x' + \frac{u'}{R}\right]
\end{aligned}
\qquad [40]
$$

where D is the plate modulus

$$D = \frac{Eh^3}{12(1-\nu^2)} \qquad [41]$$

Substituting the stress displacement relations [40] into the stress equations of motion, displacement equations of motion are obtained, which may be put in the form

$$\left[D\frac{\partial^2}{\partial x^2}-\kappa^2 Gh-\rho I\frac{\partial^2}{\partial t^2}\right]\psi_x-\left[\kappa^2 Gh\frac{\partial}{\partial x}\right]w+\left[\frac{D}{R}\frac{\partial^2}{\partial x^2}-\frac{\rho I}{R}\frac{\partial^2}{\partial t^2}\right]u=0$$

$$-\left[\kappa^2 Gh\frac{\partial}{\partial x}\right]\psi_x+\left[\frac{E_p}{R^2}+\frac{D}{R^4}-\kappa^2 Gh\frac{\partial^2}{\partial x^2}+\rho h\frac{\partial^2}{\partial t^2}\right]w+\left[\frac{E_p\nu}{R}\frac{\partial}{\partial x}\right]u=0$$

$$\left[\frac{D}{R}\frac{\partial^2}{\partial x^2}-\frac{\rho I}{R}\frac{\partial^2}{\partial t^2}\right]\psi_x+\left[\frac{E_p\nu}{R}\frac{\partial}{\partial x}\right]w+\left[E_p\frac{\partial^2}{\partial x^2}-\rho h\frac{\partial^2}{\partial t^2}\right]u=0 \qquad [42]$$

I is the moment of inertia $\frac{h^3}{12}$,and E_p is the plate compressional modulus,

$$E_p = Eh/(1-\nu^2)$$

This form is convenient in the discussion of the physical significance of the terms influencing the motion. There are three stiffnesses involved, D representing the influence of flexural, E_p of compressional and G of transverse shear stiffness, respectively. In addition, there are three inertia terms involved, ρI representing the influence of rotatory inertia, ρh in the second equation representing transverse inertia and ρh in the third equation representing longitudinal inertia. The rotation ψ_x is coupled to the transverse displacement w through the transverse shear stiffness and to the longitudinal displacement u through flexural stiffness and rotatory inertia. The coupling between displacements u and w is, essentially, both a Poisson's ratio, and a curvature effect.

For a thin shell, terms with D/R and I/R may be omitted, because D and I are proportional to the cube of the shell thickness h . As is seen from equations [42], the suppression of influence of flexural stiffness and rotatory inertia implies the assumption of a thin shell, but not vice versa, because in the first bracket of the first equation [42], D-and I-terms are unaffected by the curvature $1/R$. For a thin shell we have then

$$\left[D\frac{\partial^2}{\partial x^2} - \kappa^2 Gh - \rho I\frac{\partial^2}{\partial t^2}\right]\psi_x - \left[\kappa^2 Gh\frac{\partial}{\partial x}\right]w = 0$$

$$\left[-\kappa^2 Gh\frac{\partial}{\partial x}\right]\psi_x + \left[\frac{E_p}{R^2} - \kappa^2 Gh\frac{\partial^2}{\partial x^2} + \rho h\frac{\partial^2}{\partial t^2}\right]w + \left[\frac{E_p\nu}{R}\frac{\partial}{\partial x}\right]u = 0$$

$$\left[\frac{E_p\nu}{R}\frac{\partial}{\partial x}\right]w + \left[E_p\frac{\partial^2}{\partial x^2} - \rho h\frac{\partial^2}{\partial t^2}\right]u = 0 \qquad [43]$$

and there is no direct coupling between ψ_x and u. Going one step further, the shell can be made to approach a flat plate, that is $R \to \infty$. The first two equations reduce then to Mindlin's plate theory (17) and the last, not coupled to the first two, is the classical equation for compressional motions in a plate.

Wishing to suppress, in the more complete equations [42], the influence of transverse shear, we may not set the terms with G simply equal to zero in those equations, because these terms represent the transverse shear force, while we wish to neglect transverse shear deformation, which is effected by setting

$$\psi_x = -w' \qquad [44]$$

There will be thus only two displacement components left and the terms with G and ψ_x may be eliminated with the aid of the above relation [44]. The result is

$$\left[D\frac{\partial^4}{\partial x^4} + \frac{D}{R^4} + \frac{E_p}{R^2} + \rho h\frac{\partial^2}{\partial t^2} - \rho I\frac{\partial^4}{\partial x^2\partial t^2}\right]w + \left[-\frac{D}{R}\frac{\partial^3}{\partial x^3} + \frac{E_p\nu}{R}\frac{\partial}{\partial x} + \frac{\rho I}{R}\frac{\partial^3}{\partial x\partial t^2}\right]u = 0$$

$$\left[-\frac{D}{R}\frac{\partial^3}{\partial x^3} + \frac{E_p\nu}{R}\frac{\partial}{\partial x} + \rho\frac{I}{R}\frac{\partial^3}{\partial x\partial t^2}\right]w + \left[E_p\frac{\partial^2}{\partial x^2} - \rho h\frac{\partial^2}{\partial t^2}\right]u = 0 \qquad [45]$$

In these equations, the stiffness terms are those of the bending theory of shells, see for example Flügge (20) or Vlasov (2).

For thin shells, $D/R = I/R = 0$, they reduce to

$$\left[D\frac{\partial^4}{\partial x^4} + \frac{E_p}{R^2} + \rho h\frac{\partial^2}{\partial t^2} - \rho I\frac{\partial^4}{\partial x^2 \partial t^2}\right]w + \left[\frac{E_p \nu}{R}\frac{\partial}{\partial x}\right]u = 0$$
$$\left[\frac{E_p \nu}{R}\frac{\partial}{\partial x}\right]w + \left[E_p\frac{\partial^2}{\partial x^2} - \rho h\frac{\partial^2}{\partial t^2}\right]u = 0 \qquad [46]$$

If the rotatory inertia term is neglected and the transition to a flat plate is performed $(R \to \infty)$, the two motions in w and u become again uncoupled. The motion of w will be governed by the classical bending theory of plates and the motion of u will be again that of classical compressional theory.

If, instead, the rotatory inertia term and the flexural stiffness term are neglected in equations [46], the result is

$$\left[\frac{E_p}{R^2} + \rho h\frac{\partial^2}{\partial t^2}\right]w + \left[\frac{E_p \nu}{R}\frac{\partial}{\partial x}\right]u = 0$$
$$\left[\frac{E_p \nu}{R}\frac{\partial}{\partial x}\right]w + \left[E_p\frac{\partial^2}{\partial x^2} - \rho h\frac{\partial^2}{\partial t^2}\right]u = 0 \qquad [47]$$

which represents the familiar membrane theory of shells.

WAVE SOLUTIONS

We consider now the propagation of free harmonic waves and seek solutions of shell equations in the form

$$\psi_x(x,t) = \Psi e^{i(\omega t - \alpha x)}$$
$$w(x,t) = W e^{i(\omega t - \alpha x)} \qquad [48]$$
$$u(x,t) = U e^{i(\omega t - \alpha x)}$$

Designating by L the wave length, the natural frequency ω and the wave number α , are related to the phase velocity C by

$$\omega = \frac{2\pi C}{L} \; ; \qquad \alpha = \frac{2\pi}{L} \tag{49}$$

Introducing, further, the notation

$$\begin{aligned} m &= h/R \\ s &= C/C_s \\ \delta &= h/L \\ N &= 1/(1-\nu) \end{aligned} \tag{50}$$

and substituting forms [48] into the most complete shell equations [42] of the present investigation, the characteristic equation is calculated to be

$$\begin{aligned} &\frac{1}{3}\delta^2(2N-s^2)\left[4\pi^2\kappa^2\delta^2 - 4\pi^2 s^2\delta^2 + 2Nm^2\left(1+\frac{m^2}{12}\right)\right]\left(1-\frac{m^2}{12}\right) \\ &+ (2N-s^2)\left[\frac{2N}{\pi^2}\kappa^2 m^2\left(1+\frac{m^2}{12}\right) - 4\kappa^2 s^2\delta^2 - \frac{4}{3}\nu\kappa^2 N\delta^2 m^2 \right. \\ &\qquad\qquad \left. - \frac{4}{3}\nu^2 N^2\delta^2 m^2\right] \\ &- \left[4\frac{N^2\nu^2}{\pi^2}\kappa^2 m^2\right] \quad = \quad 0 \end{aligned} \tag{51}$$

There are thus three roots for S^2, but attention will be given only to the smallest one. In the whole range $0<\delta<\infty$, the values obtained differed too little from those obtained by the solution of three-dimensional equations, to be discerned on the scale of Fig. 1. Thus, curve 1 in Fig. 1 represents also a solution of equation [51]. Numerically largest discrepancies between the two solutions were obtained in the neighborhood of the minimum value, which may be due to a coupling effect with purely radial vibrations.

The limiting velocities may be evaluated directly from equation [51]. For very long waves, $\delta \to 0$, the limiting velocity is

$$s^2 = \left[2(1+\nu) + m^2/6(1-\nu)\right] / \left(1+\frac{m^2}{12}\right)$$

If m is small as compared to unity,

$$S^2 = 2(1+\nu)$$

which coincides with the exact solution, given by equation [23].

For very short waves, $\delta \to \infty$, equation [51] reduces to

$$(2N - S^2)^2 (K^2 - S^2) = 0 \qquad [52]$$

whatever the shell parameter m.

The double root $S^2 = 2N$ belongs to the two higher modes and will not be discussed presently. The root $S^2 = K^2$ belongs to the lower mode and the coefficient K may be now determined, precisely as in Mindlin's plate theory (17), that is by matching the velocity obtained above, with the velocity obtained from a solution of the three-dimensional theory, given by equation [21]. An appropriate value of K^2 is therefore the lowest root of

$$\sqrt{(1-K^2)(n^2-K^2)} = n\left(\tfrac{1}{2}K^2 - 1\right)^2 \qquad 0 < K < 1 \qquad [53]$$

Thus, K^2 depends upon Poisson's ratio and ranges from $K^2 = 0.76$ for $\nu = 0$ to $K^2 = 0.91$ for $\nu = 0.5$. For $\nu = 0.3$, $K^2 = 0.86$.

Letting $m \to 0$, the shell approaches a flat plate and equation [51] reduces in part, to the one obtained by Mindlin (17),

$$\frac{\pi^2\delta^2}{3}(2N - S^2)\left(\frac{1}{S^2} - \frac{1}{K^2}\right) = 1 \qquad [54]$$

which gives for long wave lengths, $\delta \to 0$

$$S^2 = \frac{2\pi^2 N}{3}\delta^2$$

that is $S^2 = 0$ for $\delta^2 = 0$, and yields thus a different limiting value than for the shell. For short wave lengths, $\delta \to \infty$, the shell equation [51] and the plate equation [54], reduce to the same expression, equation [52].

Neglecting Rotatory Inertia

Omitting now the terms with ρI in equations of motion [42], the wave solutions [48] lead to the characteristic equation,

$$\begin{aligned}(2N - s^2)\Big[\tfrac{8}{3} NK^2\pi^4\delta^4 - \tfrac{8}{3} N\pi^4\delta^4 s^2 - 4\pi^2 K^2 s^2 \delta^2 \\ + \tfrac{4}{3} N^2\pi^2\delta^2 m^2\left(1+\tfrac{m^2}{12}\right) + 2NK^2m^2\Big] \qquad [55] \\ + \Big[\tfrac{4}{9} N^2\pi^4\delta^4 m^2 (s^2 - K^2) - \tfrac{2}{9} N^3\pi^2\delta^2 m^4\left(1+\tfrac{m^2}{12}\right) \\ - \tfrac{8}{3} N^2\nu K^2\pi^2\delta^2 m^2 - 4\nu^2N^2K^2m^2 - \tfrac{8}{3}\nu^2N^3\pi^2\delta^2 m^2\Big] = 0\end{aligned}$$

The solution of equation [55] is plotted as curve 2 in Fig. 1. For long waves, $\delta \to 0$, $S^2 = 2(1+\nu)$, and for short waves, $\delta \to \infty$, $S^2 = K^2$. For a flat plate, $m = 0$, and equation [55] reduces, in part, again to the one discussed by Mindlin (17), in the corresponding case

$$S^2 = \frac{2\pi^2\delta^2}{3(1-\nu)}\left[1 + \frac{2\pi^2}{3K^2(1-\nu)}\delta^2\right]^{-1} \qquad [56]$$

Neglecting Shear

Using equations of motion [45], the membrane effect, flexural stiffness and rotatory inertia are included. The motion will consist of just two modes and the characteristic equation is obtained

$$(2N-s^2)\Big[-\tfrac{8}{3}\pi^4 N\delta^4 + \tfrac{2}{9}\pi^4 N\delta^4 m^2 + \tfrac{4}{3}\pi^4 s^2\delta^4 - \tfrac{1}{9}\pi^4 m^2 s^2\delta^4 + 4\pi^2 s^2\delta^2 + \tfrac{2}{3}N\nu\pi^2 m^2\delta^2 - 2Nm\Big(1+\frac{m^2}{12}\Big)\Big] + 2\nu N m^2\Big[\tfrac{2}{3}\pi^2 N\delta^2 - \frac{\pi^2}{3}s^2\delta^2 + 2\nu N\Big] = 0 \qquad [57]$$

The solution of equation [57] is plotted as curve 3 in Fig. 1. For long wave lengths, $\delta \to 0$, the lower velocity is

$$s^2 = \left[2(1+\nu) + m^2/6(1-\nu)\right] \Big/ \Big(1+\frac{m^2}{12}\Big)$$

For short wave lengths, $\delta \to \infty$, the lower wave velocity will be given by

$$s^2 = \frac{2}{1-\nu}$$

Which is identical to the one obtained by Mindlin (17) and is much too large as compared to the Rayleigh wave velocity. If $m = 0$, Mindlin's (17) corresponding result for the plate is obtained

$$s^2 = \frac{2\pi^2}{3(1-\nu)}\,\delta^2\Big[1+\frac{\pi^2}{3}\delta^2\Big]^{-1} \qquad [58]$$

Bending Theory

The bending theory of shells, equations [45], with $I = 0$ leads to the characteristic equation

$$(2N-s^2)\Big[4\pi^2 s^2\delta^2 - 2Nm^2\Big(1+\frac{m^2}{12}\Big) - \frac{8}{3}N\pi^4\delta^4\Big] + 4N^2m^2\Big(\nu+\frac{\pi^2\delta^2}{3}\Big)^2 = 0 \qquad [59]$$

The solution of equation [59] is plotted as curve 4 in Fig. 1. For long wave lengths, $\delta \to 0$, the lower limiting velocity is

$$S^2 = \left[2(1+\nu) + m^2/6(1-\nu)\right] \Big/ \left(1+\frac{m^2}{12}\right)$$

For short wave lengths, $\delta \to \infty$,

$$S^2 \to \frac{2}{1-\nu} - \frac{m^2}{6(1-\nu)}$$

With $m=0$, equation [59] simplifies to

$$S^2 = \frac{2\pi^2}{3(1-\nu)}\delta^2 \qquad [60]$$

which is a result of the classical plate theory.

Membrane Theory

The classical membrane theory of shells, equations [47] gives rise to the characteristic equation

$$2\pi^2 S^4 \delta^2 - 4N\pi^2 S^2 \delta^2 - NS^2 m^2 + 2Nm^2(1+\nu) = 0 \qquad [61]$$

which still contains two modes.

The solution of equation [61] is plotted as curve 5 in Fig. 1. For long waves, $\delta \to 0$, the lower mode has a velocity $S^2 = 2(1+\nu)$. For short waves, $\delta \to \infty$, the lower mode has a vanishing velocity. For a flat plate, $m = 0$, equation [61] reduces to, for any wave length,

$$S^2 = 2N$$

or

$$c^2 = E/\rho(1-\nu^2)$$

which is the compressional wave velocity in a plate. This is the only mode propagated with a finite velocity. It is, however, the mode which for $m \neq 0$ is associated with a higher wave velocity.

Thickness - Shear Motion

As an alternate way of determining the constant κ^2 in plate theory, Mindlin (17) considered the thickness-shear vibration and determined κ^2 by matching frequencies obtained by exact and approximate theories. Precisely the same can be done for a cylindrical shell.

Setting in the present theory

$$u = w = 0 \qquad \psi_x = A e^{i\omega t}$$

in equations of motion [42], the frequency ω is obtained as

$$\omega = \kappa c_s \sqrt{12}/h \qquad [63]$$

which is independent of the radius of curvature R.

Equating the right-hand sides of equations [31] and [63] results in

$$\kappa^2 = \frac{\pi^2}{12} \qquad [64]$$

The difference between κ^2 given by equations [53] and [64] is not large and vanishes if Poisson's ratio $\nu = 0.176$.

Discussion of Results

The essential results of the present study are visible from Fig. 1. The important conclusion which can be drawn is to the effect that it is possible to deduce a relatively simple shell theory, which predicts for the full range of wave lengths practically the same phase velocities of propagated waves (or frequencies of free vibration), as the three-dimensional theory.

Furthermore, it is seen, that for large wave lengths, $\delta < 0.035$, the elementary membrane shell theory yields, as expected, good results. From this value on, that is if the wave length L is smaller than the mean radius of the shell R, (since $\delta = \frac{h}{R}\frac{R}{L}$ and $\frac{h}{R}$ was taken as $\frac{1}{30}$) the shell bending theory must be used, which predicts a minimum of the phase velocity. An approximate value for this minimum may be obtained by observing that since bending is important, in this region, the effect of axial inertia may be neglected. The minimum velocity is then found to be

$$S^2_{min.} = \frac{2m\sqrt{1-\nu^2}}{(1-\nu)\sqrt{3}} \qquad [65]$$

and the corresponding value of δ is

$$\delta_T^2 = \frac{m\sqrt{3(1-\nu^2)}}{2\pi^2} \qquad [66]$$

which gives

$$\frac{L^2}{R^2} = m\frac{2\pi^2}{\sqrt{3(1-\nu^2)}} \qquad [67]$$

The expression for the minimum velocity suggested by Junger and Rosato (16), differs markedly from the one given by equation [65]. For wave lengths larger than those given by equation [66], that is if L is smaller, than, say twice the radius R, the curvature terms are practically of no importance and the shell behaves like a flat plate, but rotatory inertia and, much more so, transverse shear deformation effects, become very significant.

In summarizing, it may be stated that a thin, or medium thick, shell, behaves radically different in propagating a harmonic wave, depending upon whether the wave length is smaller or larger than the "transition" length given by equation [67]. For wave lengths larger than the transition length, the shell may be well described by the membrane theory, because all bending effects are negligible and curvature terms are predominant. For wave lengths smaller than the transition length, the shell behaves like a flat plate, since curvature terms become negligible. Thus it appears that actually in the problem on hand, no advanced shell theory is necessary. The full range may be described by the membrane shell theory (for large wave lengths) and by the Mindlin plate theory (for short wave lengths).

The existence of a minimum may be made plausible physically by observing that as the wave length becomes smaller, the contribution of radial displacements to the motion becomes larger and therefore the coupling with purely radial vibrations becomes stronger, exhibiting in the minimum a certain resonance effect. After a further decrease of wave length, the motion becomes essentially flexural and the coupling gets weaker. A certain improvement of the shell theory offered in this paper may be achieved, by matching the frequency of purely radial vibrations obtained from present theory, with the one determined from three-dimensional theory (21). This would necessitate the appropriate introduction of an additional constant in the stress-displacement relations [40] of the present shell theory, which would play a role similar to the constant κ^2.

In conclusion, it should be emphasised that the above remarks are valid only for lower phase velocities and are not necessarily true for velocities of higher modes, for mode shapes, for stresses in the shell and for group velocities. Neither are they necessarily valid for non axially-symmetric motions, which shall be investigated by the authors in a forth-coming paper.

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CAPTION TO FIG. 1

Fig. 1. Axial Phase Velocity (Lowest Mode) as a Function of Wave Length.

1 : Exact Solution of Three-Dimensional Equations and Present Shell Theory (including membrane, bending, rotatory inertia and shear deformation effects).

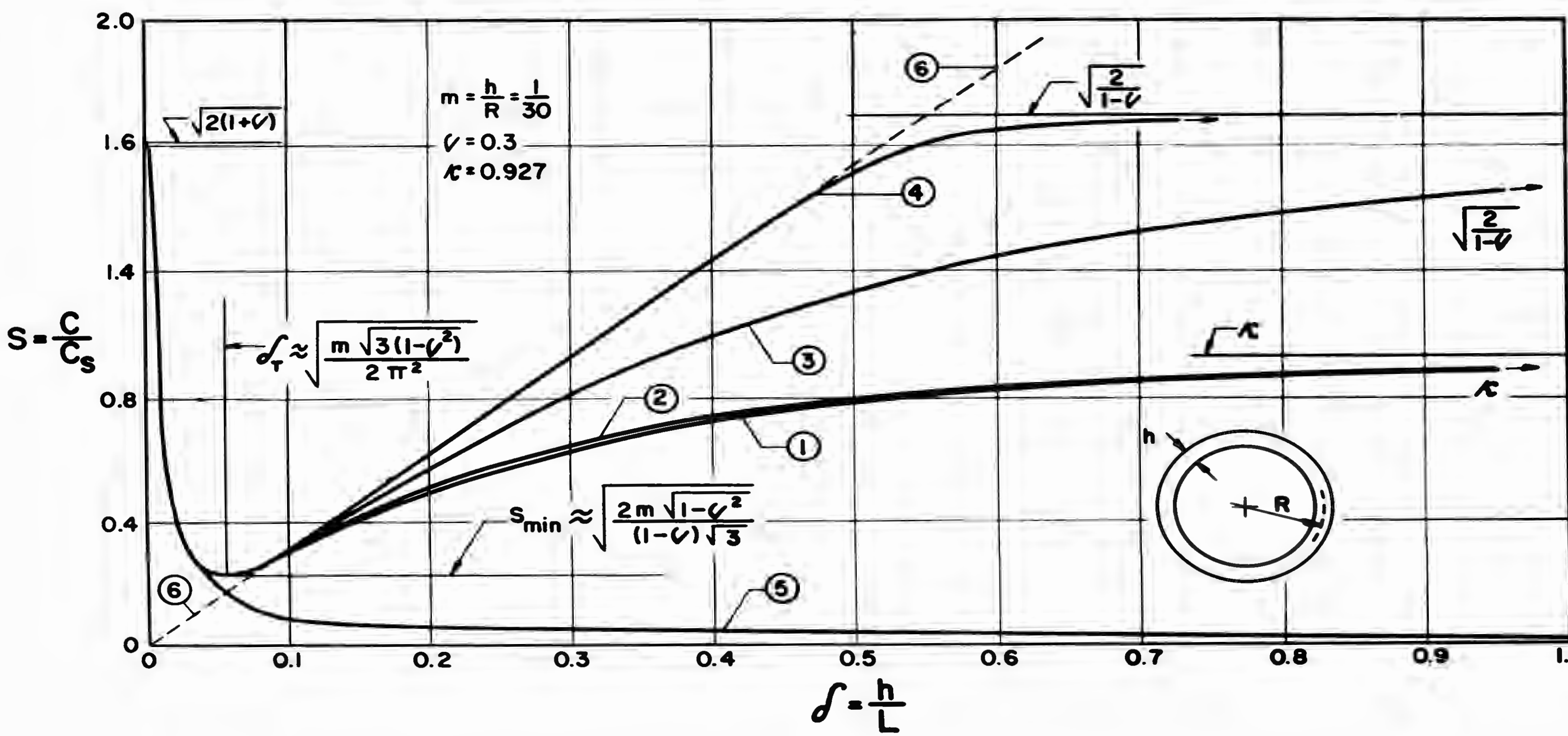
2 : Present Shell Theory, neglecting rotatory inertia effect.

3 : Present Shell Theory, neglecting shear deformation effect.

4 : Shell Bending Theory

5 : Shell Membrane Theory

6 : Classical Plate Theory



$S = \frac{C}{C_S}$
$\delta = \frac{h}{L}$
$m = \frac{h}{R} = \frac{1}{30}$
$\nu = 0.3$
$\kappa = 0.927$
$\sqrt{2(1+\nu)}$
$\delta_T \approx \sqrt{\frac{m\sqrt{3(1-\nu^2)}}{2\pi^2}}$
$S_{min} \approx \sqrt{\frac{2m\sqrt{1-\nu^2}}{(1-\nu)\sqrt{3}}}$
$\sqrt{\frac{2}{1-\nu}}$
$\sqrt{\frac{2}{1-\nu}}$
κ
κ
h
R
1
2
3
4
5
6
6
2.0
1.6
1.4
0.8
0.4
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

Fig. 1

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12. Abstract: The frequency (or phase velocity) of axially symmetric free vibrations in an elastic, isotropic circular cylinder of medium thickness is studied on the basis of the three-dimensional linear theory of elasticity and also on the basis of several different shell theories. To be in good agreement with the solution of the three-dimensional equations for short wave lengths, an approximate theory has to include the influence of rotatory inertia and transverse shear deformation for example in a manner similar as in Mindlin's plate theory. A shell theory of this (Timoshenko) type is deduced from the three-dimensional elasticity theory. From a comparison of phase velocities it appears that, to a good approximation, membrane and curvature effects on one hand and flexural, rotatory inertia and shear deformation effects, on the other hand, are mutually exclusive in two ranges of wave lengths, separated by a "transition" wave length. Thus, in the full range of wave lengths, the associated lowest phase velocities may be determined on the basis of the membrane shell theory (for wave lengths larger than the transition wave length) and on the basis of Mindlin's plate theory (for wave lengths smaller than the transition wave length).

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